

Chiral-Yang-Mills theory, non commutative differential geometry, and the need for a Lie super-algebra

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ABSTRACT: In Yang-Mills theory, the charges of the left and right massless Fermions are independent of each other. We propose a new paradigm where we remove this freedom and densify the algebraic structure of Yang-Mills theory by integrating the scalar Higgs field into a new gauge-chiral 1-form which connects Fermions of opposite chiralities. Using the Bianchi identity, we prove that the corresponding covariant differential is associative if and only if we gauge a Lie-Kac super-algebra. In this model, spontaneous symmetry breakdown naturally occurs along an odd generator of the super-algebra and induces a representation of the Connes-Lott non commutative differential geometry of the 2-point finite space.

KEYWORDS: Differential and Algebraic Geometry, Gauge Symmetry, Non-Commutative Geometry, BRST Symmetry.

*Dedicated to Yuval Ne'eman in thankful memory
of an enlightening and friendly collaboration.*

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1. Introduction

The standard model of the fundamental interactions contains three kinds of particles. The vector bosons, which gauge a local Lie algebra, the massless Fermions, and the Higgs scalars. The Yang-Mills formalism describes the interactions of the vectors with all other particles and the theory is experimentally very accurate. However, there are 2 drawbacks. The left and right chiral Fermions are independent of each other, and the status of the Higgs scalars is unclear, except that we absolutely need them to break the symmetry and give a mass to the different particles. In this paper, we propose a new construction where the Higgs scalar are included in the geometric connection one-form and prove that the corresponding covariant differential is associative if and only if this densified field is valued in the adjoint representation of a Lie-Kac super-algebra and the Fermions are graded by their chirality.

Although in essence the chiral-Yang-Mills theory is a generalization of the conventional theory, we prefer the word densification because we increase the number of algebraic relations between the known particles, and hence constrain more tightly their quantum numbers, without introducing any extra field. In this sense, our model is more economical than super-symmetry, grand unified theories or super-strings.

Our new construction kills two birds with one stone, it unexpectedly streamlines the incorporation of the 2-point finite non commutative differential geometry of Connes and

Lott [2, 3] into Yang-Mills theory and provides, at long last, a theoretical framework for the $SU(2/1)$ phenomenological classification of the elementary particles, pioneered in 1979 by Ne'eman and Fairlie [8, 6, 9].

The paper starts with the definition of the building blocks of our new construction. In section 2, we recall the construction of the Yang-Mills connection. In section 3, we summarize the geometrical construction of the BRS differential. In section 4, we recall the Connes-Lott definition of the non commutative differential geometry of the 2-point finite space, and then prove that the associated differential is algebraic. Then, in section 5, we show how to modify the Yang-Mills connection and include the Higgs field in a densified chiral-Yang-Mills gauge field connecting the Fermions of opposite chiralities and then prove that the corresponding covariant differential is associative if and only if this densified field is valued in the adjoint representation of a Lie-Kac super-algebra. In section 6, we show that the chiral-Yang-Mills model naturally integrates the non commutative differential geometry of the 2-point finite space, but contradicts the Connes-Lott analytic presentation [2, 3]. The differences are discussed in section 7: we base our construction on the structure of a finite dimensional classical Lie super-algebra, the $SU(m/n)$, $Osp(m/n)$, $D(2/1;\alpha)$, $F(4)$ and $G(3)$ series of Kac, whereas Connes and Lott rely on the very different notion of a K-cycle over an associative algebra, which forces them to deviate from the usual formalism of Quantum Field Theory and invoke the Dixmier trace. In section 8, we generalize the Yang-Mills topological Lagrangian $F \wedge F$ to the chiral case and show that it has the same symmetry as the super-Killing metric of the underlying super-algebra.

We will discuss the application of this formalism to the $SU(2/1)$ model in a separate paper.

2. The usual Yang-Mills connection one form

Everything in this section is well known, however this presentation will allow us to highlight the crucial steps of the standard construction of Yang-Mills theory to use it later as a reference when we generalize the theory. Consider a set of m left and n right spinors

$$\Psi = (\psi_L^\alpha, \psi_R^{\bar{\alpha}}), \quad \alpha = 1, 2 \dots m; \quad \bar{\alpha} = 1, 2 \dots n. \quad (2.1)$$

Each ψ spinor is a 2-dimensional column vector. The α and $\bar{\alpha}$ indices enumerate the (weak, electric, color) charges. We now consider the eight Hermitian Pauli matrices σ_μ and $\bar{\sigma}_\nu$, $\mu, \nu = 0, 1, 2, 3$ which map the right on the left spinors and vice-versa

$$\sigma : \psi_R \rightarrow \psi_L, \quad \bar{\sigma} : \psi_L \rightarrow \psi_R. \quad (2.2)$$

Notice that we can multiply σ by $\bar{\sigma}$, but we cannot multiply σ by σ or $\bar{\sigma}$ by $\bar{\sigma}$ because the left and right spinor spaces are different. We can choose a base where the Pauli matrices read:

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \bar{\sigma}_0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \bar{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \bar{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \bar{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (2.3)$$

These matrices satisfy the Dirac relation

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2g_{\mu\nu} \mathbb{1}, \quad \mu, \nu = 0, 1, 2, 3, \quad (2.4)$$

where $g_{\mu\nu}$ is the Minkowski metric $diagonal(-1, 1, 1, 1)$. The J^L and J^R anti-symmetrized products of the σ

$$\begin{aligned} J_{\mu\nu}^L &= \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \\ J_{\mu\nu}^R &= \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu), \end{aligned} \quad (2.5)$$

give us two Hermitian conjugated irreducible representations of the $SO(1, 3)$ Lorentz rotation Lie algebra: $J_R = (J_L)^\dagger$. Working in Minkowski rather than in Euclidean space is crucial if we want to consider an unequal number of left and right spinors. Indeed, in Minkowski space both the σ and the $\bar{\sigma}$ matrices are Hermitian $\sigma^\dagger = \sigma$; $\bar{\sigma}^\dagger = \bar{\sigma}$, whereas in Euclidean space σ_E and $\bar{\sigma}_E$ are Hermitian conjugates of each other $\sigma_E^\dagger = \bar{\sigma}_E$. In physical terms, the decoupling of the left and the right Fermions and the fact that the photon has only 2, and not 4, polarization states only happens for massless particles defined in Minkowski space. At the speed of light, the Lorentz contraction kills time and the direction of propagation. It only leaves a 2-dimensional transverse space to describe the spin of the massless particles.

In Yang-Mills theory, it is further assumed that the left and right spinors fall into independent (possibly reducible) representations of a Lie algebra \mathcal{A}_0 , given by the block diagonal matrices

$$\Lambda_a = \begin{pmatrix} \lambda_{a\beta}^\alpha & 0 \\ 0 & \bar{\lambda}_{a\beta}^\alpha \end{pmatrix} \quad (2.6)$$

which are closed under commutation

$$[\Lambda_a, \Lambda_b] = \Lambda_a \Lambda_b - \Lambda_b \Lambda_a = f_{ab}^c \Lambda_c \quad (2.7)$$

and satisfy the Jacobi identity

$$[\Lambda_a, [\Lambda_b, \Lambda_c]] + [\Lambda_c, [\Lambda_a, \Lambda_b]] + [\Lambda_b, [\Lambda_c, \Lambda_a]] = 0 \quad (2.8)$$

The f_{bc}^a are complex numbers called the structure constants of the Lie algebra. Let us now introduce the Cartan exterior differential d . Let x^μ , $\mu = 0, 1, 2, 3$ denote a local system of coordinates. We define the symbols dx^μ and assume that they commute with the coordinates but anti-commute among themselves:

$$x^\mu x^\nu = x^\nu x^\mu, \quad x^\mu dx^\nu = dx^\nu x^\mu, \quad dx^\mu dx^\nu = -dx^\nu dx^\mu. \quad (2.9)$$

A polynome of degree p in the dx^μ is called a p -form. The Cartan exterior differential is defined in terms of the partial derivatives relative to the coordinates as $d = dx^\mu \partial/\partial x^\mu$. Since the dx^μ anti-commute and the partial derivatives commute, the Cartan differential satisfies the fundamental rule

$$d^2 = 0. \quad (2.10)$$

The Yang-Mills connection A is defined as a space-time dependent Cartan differential one-form valued in the adjoint representation of the Lie algebra

$$A = dx^\mu A_\mu^a(x) \Lambda_a. \tag{2.11}$$

Notice that A is a matrix-valued one-form, but $A_\mu^a(x)$ is just a set of commuting complex valued functions. Using A , we define the covariant exterior differential

$$D = d + A. \tag{2.12}$$

D acts separately on the left and right spinors, $d = dx^\mu \partial_\mu$ acts as a derivative and A acts by matrix multiplication.

$$D\psi_L^\alpha(x) = dx^\mu (\partial_\mu \psi_L^\alpha(x) + A_\mu^a(x) \lambda_{a\beta}^\alpha \psi_L^\beta(x)). \tag{2.13}$$

We now compute $DD\psi$. The facts that the dx^μ anti-commute and the ∂_μ satisfy the Leibnitz rule imply that $DD\psi$ does not depend on the derivatives of ψ . For this reason DD is not a derivation but an algebraic 2-form called the curvature 2-form F

$$DD\psi = F \psi \quad \text{with} \quad F = dA + AA. \tag{2.14}$$

If we develop all the indices in the AA term we find

$$AA = dx^\mu dx^\nu A_\mu^a A_\nu^b \Lambda_a \Lambda_b. \tag{2.15}$$

Since the dx^μ anti-commute and the A_μ^a commute, this expression is anti-symmetric in the (a, b) indices and therefore only involves the commutator of the Λ_a matrices. Using the closure of the Lie algebra, we can rewrite AA as a linear combination of the Λ_a matrices, and see that F is also valued in the adjoint representation of the Lie algebra \mathcal{A}_0 .

$$F = dA + AA = \left(dA^c + \frac{1}{2} A^a A^b f_{ab}^c \right) \Lambda_c = F^c \Lambda_c. \tag{2.16}$$

The definition $DD = F$ only makes sense if the product of 3 or more D is associative:

$$DDD\psi = (DD)D\psi = D(DD)\psi \iff FD\psi = D(F\psi). \tag{2.17}$$

These equations are equivalent to the Bianchi identity:

$$dF + [A, F] = 0. \tag{2.18}$$

Expanding out all the indices, the terms in $A dA$ cancel because the 2-form dA^a commutes with the 1-form A^b . However the term trilinear in A

$$dx^\mu dx^\nu dx^\rho A_\mu^a A_\nu^b A_\rho^c [\Lambda_a, [\Lambda_b, \Lambda_c]] \tag{2.19}$$

vanishes if and only if the Λ matrices satisfy the Jacobi identity, that is, if and only if the connection A is valued in the adjoint representation of a Lie algebra.

Theorem 1. *The Yang-Mills covariant differential $D = d + A$ is associative if and only if the connection 1-form A is valued in the adjoint representation of a Lie algebra.*

$$D \text{ is associative} \iff \text{Bianchi} \iff \text{Jacobi}. \tag{2.20}$$

This theorem is implicit in the original paper of Yang and Mills [17], and possibly even in the much earlier work of Elie Cartan, but I do not know where it was first stated explicitly.

3. The BRS differential

The material in this section is also known, but will be needed in section 6. The BRS differential s controls the renormalization process of the standard Yang-Mills theory [1]. The most economical way to introduce the BRS operator and the Faddeev Popov [5] ghost field c is to extend the differential d (2.10) and the Yang-Mills connection 1-form A (2.11) as

$$\check{d} = d + s, \quad \check{A} = A + c. \tag{3.1}$$

We assume that the c ghost anti-commutes with all 1-forms and we construct once more as in (2.12), (2.14) the covariant differential \check{D} and the curvature 2-form \check{F} :

$$\check{D} = \check{d} + \check{A}, \quad \check{F} = \check{d}\check{A} + \check{A}\check{A}. \tag{3.2}$$

We now postulate the Maurer-Cartan horizontality condition:

$$\check{F} = F. \tag{3.3}$$

This equation must be understood as a constraint which defines the action of s on A and c . Substituting (2.14) and (3.2) in (3.3), we find:

$$sA = -dc - [A, c], \quad sc = -\frac{1}{2}[c, c]. \tag{3.4}$$

If we write explicitly the Λ Lie algebra matrices as in (2.11), we see that (3.4) includes only the commutator of the Λ matrices (2.7) and can be written as

$$sA^a = -dc^a - f_{bc}^a A^b c^c, \quad sc^a = -\frac{1}{2} f_{bc}^a c^b c^c. \tag{3.5}$$

We define the action of s on higher forms by the condition $sd + ds = 0$. We may then verify that the Jacobi identity implies the consistency condition $s^2 = 0$.

Theorem 2. *The Yang-Mills-Faddeev-Popov covariant differential $\check{D} = \check{d} + \check{A}$ is associative if and only if the connection 1-form \check{A} is valued in the adjoint representation of a Lie algebra, and if so the BRS operator s is nilpotent.*

$$\text{Bianchi} \iff \text{Jacobi} \iff s^2 = 0. \tag{3.6}$$

This theorem is implicit in the work of Becchi, Rouet and Stora [1], who only recognized the variation sc as the parallel transport on a Lie group. To the best of my knowledge, the complete identification of the 2 BRS equations with the horizontality conditions of Cartan was first proposed in my own work [13].

4. The Connes-Lott 2-points non commutative differential algebra

This section recalls our last building block. In their seminal paper [2, 3], Connes and Lott introduced the non commutative differential geometry of a discrete space with just 2 points. They consider 2 complementary projectors:

$$\epsilon + \bar{\epsilon} = 1, \quad \epsilon\bar{\epsilon} = \bar{\epsilon}\epsilon = 0, \quad \epsilon\epsilon = \epsilon, \quad \bar{\epsilon}\bar{\epsilon} = \bar{\epsilon}, \tag{4.1}$$

and define their formal differentials $\delta\epsilon$ and $\delta\bar{\epsilon}$. The interesting observation is that the Leibnitz rule, together with the properties of the projectors, imply non commutativity of the resulting differential algebra:

$$\delta(\epsilon + \bar{\epsilon}) = \delta(1) = 0 \Rightarrow \delta\bar{\epsilon} = -\delta\epsilon, \quad (4.2)$$

$$(\epsilon + \bar{\epsilon}) \delta\epsilon = \delta\epsilon = \delta(\epsilon^2) = \delta\epsilon \epsilon + \epsilon \delta\epsilon \Rightarrow \bar{\epsilon} \delta\epsilon = \delta\epsilon \epsilon, \quad (4.3)$$

$$(\epsilon + \bar{\epsilon}) \delta\bar{\epsilon} = \delta\bar{\epsilon} = \delta(\bar{\epsilon}^2) = \delta\bar{\epsilon} \bar{\epsilon} + \bar{\epsilon} \delta\bar{\epsilon} \Rightarrow \epsilon \delta\bar{\epsilon} = \delta\bar{\epsilon} \bar{\epsilon}, \quad (4.4)$$

i.e., commuting with $\delta\epsilon$ exchanges the 2 projectors. In this formalism, a function f is a vector in the complex algebra with generators ϵ and $\bar{\epsilon}$:

$$f = g \epsilon + h \bar{\epsilon}, \quad (4.5)$$

a connection 1-form ω is an object linear in $\delta\epsilon$:

$$\omega = u \epsilon \delta\epsilon + v \bar{\epsilon} \delta\bar{\epsilon}, \quad (4.6)$$

and the corresponding curvature F can be written as:

$$F = \delta\omega + \omega\omega = (u - v + uv) \delta\epsilon \delta\bar{\epsilon}. \quad (4.7)$$

The curvature vanishes for the Cartan connection:

$$\omega_0 = 2 \epsilon \delta\epsilon - 2 \bar{\epsilon} \delta\bar{\epsilon}. \quad (4.8)$$

Up to here, we exactly followed [3], but from now on our treatment is original. These equations look more familiar if we introduce the chirality operator χ :

$$\chi = \epsilon - \bar{\epsilon} \Rightarrow \chi^2 = 1, \quad \chi\epsilon = \epsilon\chi, \quad \chi\delta\chi = -\delta\chi\chi. \quad (4.9)$$

Now 1 and χ are the standard generators of the 2 elements multiplicative group and the same flat connection now takes the standard form of the Cartan left invariant 1-form:

$$\omega_0 = \chi \delta\chi = \chi^{-1} \delta\chi, \quad (4.10)$$

$$F_0 = \delta\chi\delta\chi + \chi \delta\chi \chi \delta\chi = \delta\chi\delta\chi - \delta\chi \chi \chi \delta\chi = 0. \quad (4.11)$$

Forms and functions do not commute and we can define the covariant differential Δ :

$$\Delta f = \delta f + \frac{1}{2}(\omega_0 f - f \omega_0), \quad (4.12)$$

$$\Delta \omega = \delta \omega + \frac{1}{2}(\omega_0 \omega + \omega \omega_0), \quad (4.13)$$

and so on, alternating the signs for higher forms. Expanding all terms, and using (4.5, 4.6, 4.9), we verify that, for any function f , and any p-form ω , we have:

$$\Delta f = \Delta \omega = 0. \quad (4.14)$$

In other words, the differential δ is algebraic.

5. The new Chiral-Yang-Mills connection

Consider again a set of left and right spinors $(\psi_L^\alpha, \psi_R^{\bar{\alpha}})$ as defined in (2.1). Consider the 2 by 2 matrix notation corresponding to the left-right splitting of the spinors. The top-left diagonal block is really of size mm , the bottom-right of size nn . We define the left and right projectors ϵ and $\bar{\epsilon}$ and the chirality operator χ :

$$\epsilon : \psi_L \rightarrow \psi_L, \quad \bar{\epsilon} : \psi_R \rightarrow \psi_R, \\ \epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \chi = \epsilon - \bar{\epsilon} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.1)$$

and the matrix 1-forms Γ and $\bar{\Gamma}$:

$$\Gamma = \begin{pmatrix} dx^\mu \sigma_\mu & 0 \\ 0 & dx^\mu \bar{\sigma}_\mu \end{pmatrix}, \quad \bar{\Gamma} = \begin{pmatrix} dx^\mu \bar{\sigma}_\mu & 0 \\ 0 & dx^\mu \sigma_\mu \end{pmatrix}. \quad (5.2)$$

Using (2.5), (2.9) we have:

$$\Gamma \bar{\Gamma} = dx^\mu dx^\nu (J_{\mu\nu}^L \epsilon + J_{\mu\nu}^R \bar{\epsilon}). \quad (5.3)$$

The main point of this paper is to generalize the Yang-Mills connection 1-form of equation (2.11) into the chiral-Yang-Mills connection 1-form \hat{A} :

$$\hat{A} = A + \Gamma \Phi. \quad (5.4)$$

where the new term $\Gamma \Phi$ connects the left and the right Fermions. Let the $a, b, c \dots$ alphabet labels the charges of A , and the $i, j, k \dots$ alphabet labels the charges of Φ :

$$A = A^a \Lambda_a, \quad \Phi = \Phi^i \Lambda_i. \quad (5.5)$$

In the 2 by 2 matrix notation (5.1), the Λ_a and Λ_i matrices are respectively block diagonal and anti-diagonal:

$$\Lambda_a = \begin{pmatrix} \lambda_a & 0 \\ 0 & \bar{\lambda}_a \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} 0 & \lambda_i \\ \bar{\lambda}_i & 0 \end{pmatrix}, \quad (5.6)$$

and we have

$$\Gamma \Lambda_i = \Lambda_i \bar{\Gamma}, \quad \bar{\Gamma} \Lambda_i = \Lambda_i \Gamma. \quad (5.7)$$

Let us now study the spin structure of A and Φ , i.e. the way the Lorentz index μ appears in these definitions. A is a 1-form linear in dx^μ . Explicitly:

$$A = A^a \Lambda_a = dx^\mu A_\mu^a(x) (\lambda_{a\beta}^\alpha \epsilon + \bar{\lambda}_{a\beta}^{\bar{\alpha}} \bar{\epsilon}). \quad (5.8)$$

Therefore the function $A_\mu^a(x)$ is as usual a vector field. However, we have to remember that the only way to connect a left to a right spinor of the $SO(1,3)$ Lorentz algebra is to use the Pauli matrices (2.2)–(2.4) which, by definition, carry a μ index. As a result, the Pauli matrices σ_μ steal the μ index usually carried by the gauge field and the i charge is now carried by a field without Lorentz index, i.e. by a scalar field $\Phi^i(x)$:

$$\Gamma \Phi = \Gamma \Phi^i \Lambda_i = dx^\mu \Phi^i(x) (\lambda_{i\beta}^\alpha \sigma_\mu + \bar{\lambda}_{i\beta}^{\bar{\alpha}} \bar{\sigma}_\mu). \quad (5.9)$$

Consider now the λ and $\bar{\lambda}$ matrices which act on the charges α, β or $\bar{\alpha}, \bar{\beta}$ of the spinors. In Yang-Mills theory, the Λ matrices were chosen to represent a Lie algebra. In our generalization, we make no a priori assumptions on the λ matrices, we will deduce their structure from the consistency conditions on the connection 1-form \hat{A} . At this point, $(\lambda_a, \bar{\lambda}_a, \lambda_i, \bar{\lambda}_i)$ are just defined as complex valued matrices of respective dimensions (mm, nn, mn, nm) where m and n are the numbers of left and right spinors. This was the approach of Gell-Mann when he was studying the strong interactions. He wanted to generalize the 3 λ matrices of the SU(2) (proton,neutron) isospin of Wigner and include the strange particles. He successively tried to use 4,5,6, or 7 matrices, before hitting the magic eight-fold way. Independently, Ne'eman, who had read Dynkin, started from the G_2 star of David, and then restricted it to $A_2 = \text{SU}(3)$ flavor [7].

Notice that by itself Γ (5.2) or Λ_i (5.6) cannot act on the ψ , because Γ changes the chirality of the Fermions without changing the α charges and Λ_i changes the α charges into $\bar{\alpha}$ charges without changing the chirality. Only the product $\Gamma\Lambda_i$ behaves correctly. We now construct as in (2.12) the chiral-Yang-Mills covariant differential

$$\hat{D} = d + \hat{A} = d + A + \Gamma\Phi \tag{5.10}$$

and study as in (2.14) the repeated action of \hat{D} . Again $\hat{D}\hat{D}$ is a 2-form:

$$\hat{D}\hat{D}\psi = \hat{F}\psi, \tag{5.11}$$

$$\hat{F} = d\hat{A} + \hat{A}\hat{A} = dA + AA + (A\Gamma\Phi + \Gamma\Phi A) + \Gamma\Phi\Gamma\Phi. \tag{5.12}$$

If we develop (5.12) in its (5.5) components, we find successively that

$$AA = dx^\mu dx^\nu A_\mu^a A_\nu^b \Lambda_a \Lambda_b \tag{5.13}$$

is anti-symmetric in (a, b) (2.15), so that the Λ_a matrices must be closed under commutation

$$[\Lambda_a, \Lambda_b] = \Lambda_a \Lambda_b - \Lambda_b \Lambda_a = f_{ab}^c \Lambda_c. \tag{5.14}$$

Then, in $(A\Gamma\Phi + \Gamma\Phi A)$, A and Γ are both 1-forms, and anti-commute. This produces a commutator which must close on the Λ_i :

$$A\Gamma\Phi + \Gamma\Phi A = -\Gamma A^\alpha \Phi^i (\Lambda_a \Lambda_i - \Lambda_i \Lambda_a) \Rightarrow [\Lambda_a, \Lambda_i] = f_{ai}^j \Lambda_j. \tag{5.15}$$

Finally, we consider the term $\Gamma\Phi\Gamma\Phi$. Using (5.7) we find

$$\Gamma\Phi\Gamma\Phi = \Gamma\bar{\Gamma} \Phi^i \Phi^j \Lambda_i \Lambda_j. \tag{5.16}$$

Since the scalar fields Φ^i and Φ^j fields commute, this term is symmetric in (i, j) , hence the Λ_i matrices must close under anti-commutation:

$$\{\Lambda_i, \Lambda_j\} = \Lambda_i \Lambda_j + \Lambda_j \Lambda_i = d_{ij}^a \Lambda_a, \tag{5.17}$$

i.e. the d_{ij}^a structure constants are symmetric in i, j . Again, we require as in (2.17) the associativity of the triple covariant differential

$$\hat{D}\hat{D}\hat{D}\psi = (\hat{D}\hat{D})\hat{D}\psi = \hat{D}(\hat{D}\hat{D})\psi \iff \hat{F}\hat{D}\psi = \hat{D}(\hat{F}\psi). \tag{5.18}$$

These equations are again equivalent as in (2.18) to the Bianchi identity

$$d\widehat{F} + [\widehat{A}, \widehat{F}] = 0. \quad (5.19)$$

But this time, when we expand out the term trilinear in \widehat{A} , we find that the Λ matrices must satisfy the super Jacobi identity:

$$[\Lambda_a, [\Lambda_b, \Lambda_c]] + [\Lambda_c, [\Lambda_a, \Lambda_b]] + [\Lambda_b, [\Lambda_c, \Lambda_a]] = 0, \quad (5.20)$$

$$[\Lambda_a, [\Lambda_b, \Lambda_i]] + [\Lambda_i, [\Lambda_a, \Lambda_b]] + [\Lambda_b, [\Lambda_i, \Lambda_a]] = 0, \quad (5.21)$$

$$[\Lambda_a, \{\Lambda_i, \Lambda_j\}] - \{\Lambda_j, [\Lambda_a, \Lambda_i]\} + \{\Lambda_i, [\Lambda_j, \Lambda_a]\} = 0, \quad (5.22)$$

$$[\Lambda_i, \{\Lambda_j, \Lambda_k\}] + [\Lambda_k, \{\Lambda_i, \Lambda_j\}] + [\Lambda_j, \{\Lambda_k, \Lambda_i\}] = 0. \quad (5.23)$$

In other words, the Λ_a and Λ_i matrices form a Lie-Kac super-algebra, where the a charges label the even generators of the Lie sub-algebra (5.14 and 5.20 are identical to 2.7 and 2.8) and the i charges label the odd generators.

Theorem 3. *The chiral-Yang-Mills covariant differential $\widehat{D} = d + \widehat{A}$ is associative if and only if the Fermions are graded by their chirality and the connection 1-form \widehat{A} is valued in the adjoint representation of a Lie-Kac super-algebra.*

$$\widehat{D} \text{ is associative} \iff \text{Bianchi} \iff \text{super-Jacobi}. \quad (5.24)$$

As far as we know, this theorem is new.

6. Non-commutative differential geometry of the chiral Fermions

In the chiral Fermion space, we can represent the Connes-Lott 2-point non commutative differential algebra as follows. The chirality operator (4.9) of the 2-point geometry is represented by the chirality operator (5.1) of the Fermion space. The δ differential (4.2) is represented by the super-commutator with Γ (5.2) times a fixed odd generator v of the super-algebra (5.24)

$$\delta = -[\Gamma v,], \quad \delta\psi = -\Gamma v \psi. \quad (6.1)$$

If we directly compute, in the 2 by 2 matrix notation (5.1), (5.2), the Cartan 1-form ω_0 (4.10), using our new definition of the δ variation of χ , we find

$$\omega_0 = \chi \delta\chi = 2 \Gamma v. \quad (6.2)$$

We recover as observed in (4.14) that $\Delta = \delta + \frac{1}{2}[\omega_0,]$ vanishes on v, ϵ, χ ; ω_0 has vanishing δ curvature (4.11); and δ^2 vanishes on the chirality operators χ and ϵ

$$\delta \omega_0 = -\omega_0 \omega_0 \iff \delta \Gamma v = -\{\Gamma v, \Gamma v\}, \quad \delta^2 \chi = \delta^2 \epsilon = 0. \quad (6.3)$$

Thanks to the super-Jacobi identity (5.23), δ^2 also vanishes on Γv itself:

$$\delta^2 \Gamma v = -2 [\Gamma v, \{\Gamma v, \Gamma v\}] = 0. \quad (6.4)$$

These equations look very much like the BRS equations of section 3, except that the variation of Γv does not involve the usual factor $-1/2$ found in (3.5), hence δ^2 does not vanish on ψ and we have:

$$\delta^2 \psi = \frac{1}{2} \Gamma \bar{\Gamma} \{v, v\} \psi. \quad (6.5)$$

Let us now take advantage of the similarity between s and δ . We propose to mix the chiral (5.4) and the BRS (3.1) extensions of the Yang-Mills connection 1-form (2.11) and of the corresponding differentials and to consider the new chiral-Yang-Mills-Higgs-Kibble-Faddeev-Popov-Connes-Lott connection 1-form \tilde{A}

$$\tilde{A} = \hat{A} + \hat{c} = \check{A} + \Gamma \check{\Phi}, \quad \tilde{d} = \hat{d} + s = \check{d} + \delta, \quad (6.6)$$

where we either decompose between gauge and ghost fields

$$\hat{A} = A + \Gamma \Phi, \quad \hat{c} = c + \Gamma v, \quad \hat{d} = d + \delta, \quad (6.7)$$

or between the even and odd generators of the super-algebra

$$\check{A} = A + c, \quad \check{\Phi} = \Phi + v, \quad \check{d} = d + s. \quad (6.8)$$

We now define the covariant differential \tilde{D} and the curvature \tilde{F} by the usual formulas

$$\tilde{D} = \tilde{d} + \tilde{A}, \quad \tilde{F} = \tilde{d}\tilde{A} + \tilde{A}\tilde{A}. \quad (6.9)$$

However, if we compute \tilde{D}^2 we find a term δ^2 which does not vanish (6.5). Hence, in contradistinction to (2.14), we now have $\tilde{D}^2 = \tilde{F} + \delta^2$. The Cartan horizontality condition (3.3) is really a consistency condition on D^2 , rather than on F , so the correct generalization is:

$$\tilde{D}^2 = \tilde{F} + \delta^2 = \tilde{d}\tilde{A} + \tilde{A}\tilde{A} + \delta^2 = d\hat{A} + \hat{A}\hat{A} = \hat{F} = \hat{D}^2. \quad (6.10)$$

The Cartan condition usually implies the exact gauge symmetry of the theory, so we will not be surprised later to find that this δ^2 correction controls the symmetry breaking of the standard model. This equation is again a condition which defines the action of s and δ on the gauge and ghosts fields. If we develop according to the different gradations, we find the usual definition (3.4) of s

$$sA = -dc - [A, c], \quad sc = -\frac{1}{2}[c, c], \quad s\Phi = -[c, \Phi], \quad (6.11)$$

a novel definition of the action of δ on the gauge fields

$$\delta A = -\{\Gamma v, A\} = -\Gamma[v, A], \quad \delta \Gamma \Phi = -\{\Gamma v, \Gamma \Phi\} = -\Gamma \bar{\Gamma} \{v, \Phi\}, \quad (6.12)$$

and the $s \delta$ compatibility conditions

$$\delta c = -\{\Gamma v, c\} = -\Gamma[v, c], \quad dv = sv = 0. \quad (6.13)$$

Again, we define the action of δ on higher forms by the equation $\delta d + d\delta = 0$. The action of δ on all fields is now defined algebraically. We can verify that $s^2 = 0$ as usual. However, we do not have $\delta^2 = 0$, but thanks to the super-Jacobi identity (5.20)–(5.23) we can verify that $\delta^2 = \frac{1}{2}\{\Gamma v, \Gamma v\}$ on all fields, generalizing (6.5).

The definition of δ deduced from the horizontality condition (6.10) extends to A, c and Φ (6.12)–(6.13) the geometrical condition (4.14) that Δ should vanish on all fields. In other words, we have exactly found the desired condition which usually holds true on a principal fiber bundle, the δ “vertical” geometry (6.1) exactly matches the non commutative differential geometry (4.2) of the 2-point finite space.

We also learn from (6.10) that the relevant gauge curvature 2-form is not exactly \widehat{F} but $\widetilde{F} = \widehat{F} - \delta^2$. As mentioned before, this object is also a 2-form if and only if we have the super-Jacobi identity. If we develop we find:

$$\widetilde{F} = dA + \frac{1}{2}[A, A] + \frac{1}{2}\Gamma\overline{\Gamma}(\{\Phi, \Phi\} - \{v, v\}) - \Gamma D\Phi. \quad (6.14)$$

\widetilde{F} vanishes in the “vacuum” $A = 0, \Phi = \pm v$. It is therefore natural to rewrite \widetilde{F} in terms of the shifted field $\check{\Phi} = \Phi + v$ (6.8) and obtain

$$\widetilde{F} = dA + \frac{1}{2}[A, A] + \Gamma D(\check{\Phi} - v) + \frac{1}{2}\Gamma\overline{\Gamma}(\{\check{\Phi} - v, \check{\Phi} - v\} - \{v, v\}). \quad (6.15)$$

Finally, using the definition (6.3), (6.12) of δ , we can rewrite this last equation as

$$\widetilde{F} = (d + \delta)(A + \Gamma\check{\Phi}) + (A + \Gamma\check{\Phi})(A + \Gamma\check{\Phi}) \quad (6.16)$$

which exactly matches the definition of Connes and Lott [3]. Without the need to introduce anything else besides a generalized connection 1-form, we have recovered their main result. The inclusion of the discrete differential δ in the definition of the horizontal differential yields the Higgs-Kibble-Brout-Englert spontaneous symmetry breaking pattern of the standard model.

When we consider the differential $\widehat{d} = d + \delta$ (6.7), (6.16), we actually replace the Yang-Mills geometry, where one considers a principal fiber bundle with Minkowski space as the base and the Lie group as the fiber, by the chiral-Yang-Mills geometry where the base space consists of 2 copies of Minkowski space, one populated by left movers and one by right movers, reminiscent of super-string theory.

In this presentation, we have identified the Connes-Lott discrete δ with the super-BRS operator associated to the odd generators of the super-algebra, and Γv with the corresponding super-ghost. Usually, the ghost of a tensor field has one index less than the gauge field, for example the ghost of the gauge 3-form of super-gravity is a 2-form, the ghost of the graviton is a vector, the ghost of the Yang-Mills vector is a scalar. Therefore, in the present case, where the ‘gauge field’ is a scalar Φ , it is perhaps not so surprising to find that its ghost is a constant v satisfying $dv = sv = 0$ (6.13).

Another striking fact is the interpretation of v as the vacuum expectation value of the Φ field. It may seem unusual to interpret the ghost as the vacuum value of the gauge field. But in fact, in the conventional Yang-Mills theory, the Yang-Mills-Faddeev-Popov connection $\check{A} = A + c$ (3.1) defined on the principal bundle [14] never vanishes, since its pull back on the gauge group is the left-invariant form. In the vacuum, we really have $A = 0$ but $\check{A} = c$, i.e. the vacuum expectation value of the full Yang-Mills connection is the Faddeev-Popov ghost form. Therefore, the super-case $\langle \Phi \rangle = v$ is the natural generalization of $\langle \check{A} \rangle = c$.

To summarize, we have verified that:

Theorem 4. *The new chiral covariant differential $\tilde{D} = \tilde{d} + \tilde{A}$ is associative if and only if the Fermions are graded by their chirality and the connection 1-form \tilde{A} is valued in the adjoint representation of a Lie-Kac super-algebra. If so, the BRS operator s is nilpotent and the square of the super-BRS Connes-Lott discrete differential δ is a constant 2-form.*

$$\begin{aligned} \text{Bianchi} &\iff \text{super-Jacobi} \iff \\ s^2 = s\delta + \delta s &= 0 \quad ; \quad \delta^2 = \frac{1}{2} \Gamma\bar{\Gamma} \{v, v\} . \end{aligned} \tag{6.17}$$

This theorem is new.

7. Comparison with the Connes-Lott model

The integration of the 2-point non-commutative differential geometry in the chiral-Yang-Mills theory discussed in the last section is very pleasing. It corresponds in essence to the physical interpretation that Connes and Lott give of their own work. Space time really has 2 chiral sheets, populated by left and right Fermions, and the role of the Higgs field is to connect them, and give them a mass. Superficially, the Higgs potentials computed here and in their paper agree. We both recover directly the spontaneous symmetry breakdown of the theory and subtract the constant term, so the energy of the vacuum vanishes. However, if we compare in detail the Connes-Lott construction to the present one, things become very different.

Whereas, in our new formalism, we found that a Lie super-algebra, for example $SU(2/1)$, is necessary to ensure the associativity of the covariant differential, Connes and Lott do not consider this underlying structure. Rather, they start from the universal differential envelope of an associative algebra and arrive at the Lie algebra $U(2).U(1)$ which acts separately on the 2 chiral subspaces. They do not really give a structure to the Λ_i odd matrices. The closest they come to our definition is when they consider their junk ideal J . J is needed to quotient out the symmetric product of the dx^μ , which exists in their abstract universal differential envelope, but must be removed to identify their $\pi(dx^\mu)$ with a standard exterior form. But in the interesting case describing the leptons, they find that the junk ideal only contains in the 2-form sector the terms bilinear in two Dirac matrices. The effect of quotienting by the Junk ideal therefore removes the trace of the term $\Gamma\bar{\Gamma}vv$ which appears in the curvature and lead to their formula for the Higgs potential $V(\phi) = Tr(v^2 - Tr(v^2).Id)^2((\phi - 1)^2 - 1)^2$. But since the term in v^2 occurs in the definition of the curvature, we expect it to be valued in the adjoint representation of the Lie algebra of the model. However, if we look at the quantum numbers of the leptons, we observe that the super-trace of the photon matrix vanishes, not its trace, and Weinberg has explained in his original paper [16] that if we had in the algebra a second $U(1)$ operator besides the photon, then it would be coupled to the conserved lepton number and be massless. But experimentally, this field does not exist. Therefore, by the argument of Weinberg, a traceless v^2 term cannot be part of the curvature and in fact, a close reading of their Cargese lectures [3] shows that Connes and Lott are aware of this difficulty. We

do not exactly know how to resolve this contradiction, but it may have to do with the Wick rotation. Connes and Lott work in Euclidean space where left and right Fermions are not well separated. In the end, they must rotate back to Minkowski space. But if we think about it, the trace and the super-trace only differ by a relative sign when we trace successively over the left and the right spinors. May be the Wick rotation should involve in their formalism a transmutation of the trace in to a super-trace. Notice also that in the Connes Lott construction, a rather difficult calculation is needed to find, at each degree n , the exact content of their junk ideal $J^{[n]}$. The construction is needed to insure that they can work on the physical fields, i.e. on the equivalence classes obtained after division by J . On the other hand, in our formalism, the compatibility is insured to all exterior degrees by the Bianchi identity.

On the technical side, we have completely by-passed their construction of the universal enveloping algebra, and the need to quotient by the junk ideal or to invoke the Dixmier trace. We remain strictly within the framework of standard Quantum Field Theory, gauge fields and anti-commuting connection 1-forms. In this respect, our formalism is much more economical. Although we adopt their 2-point non-commutative differential geometry, the way we incorporate it in the new chiral-Yang-Mills theory is different from their construction and leads to a very different type of constraints on the acceptable gauge groups and particle multiplets. In our construction, we must gauge a finite dimensional classical Lie super-algebra, i.e. a direct sum of members of the $SU(m/n)$, $Osp(m/n)$, $D(2/1;\alpha)$, $F(4)$ and $G(3)$ series of Kac. The Fermions must fall into finite dimensional representations of the chosen super-algebra, and be graded by chirality. The classification of the acceptable Connes-Lott models is completely different. In a nutshell, Connes and Lott had the correct physical intuition, but the wrong mathematical background!

To summarize, we found a way to import the fundamental concept of non-commutative differential geometry: the non commutative 2-point space which exchanges the 2 chiral sheets, and preserve the main result of Connes-Lott: a geometrical interpretation of the Higgs field yielding automatically the symmetry breaking Higgs potential. We did this, dropping their unusual formalism of the K -cycle, but finding as a consistency condition, the existence of an underlying Lie-Kac super-algebra which grades the Fermions by their chirality. Hence, as a cherry on the cake, we deduce that the chiral Fermion geometry implies the $SU(2/1)$ internal super-symmetry of Ne'eman and Fairlie [8, 6, 9].

8. The topological term and the super-Killing metric

As shown in section 5, the chiral-Yang-Mills curvature 2-form \widehat{F} is valued in the adjoint representation of a Lie-Kac super-algebra:

$$\widehat{F} = \widehat{F}^a \Lambda_a + \widehat{F}^i \Lambda_i, \tag{8.1}$$

$$\widehat{F}^a = dA^a + \frac{1}{2} f_{bc}^a A^b A^c + \frac{1}{2} \Gamma \bar{\Gamma} d_{ij}^a \Phi^i \Phi^j, \tag{8.2}$$

$$\widehat{F}^i = \Gamma(d\Phi^i + f_{aj}^i A^a \Phi^j). \tag{8.3}$$

Notice that we must use the equation $\Lambda_i \Gamma = \bar{\Gamma} \Lambda_i$ (5.7) before extracting the Λ_i matrices on

the right. If we introduce the super-indices M, N which span the even and odd generators of the super-algebra

$$\widehat{F} = \widehat{F}^M \Lambda_M ; \quad M = a, b, \dots, i, j, \dots \quad (8.4)$$

We can readily verify that the topological 4-form

$$\mathcal{L}_T = \widehat{g}_{MN} \widehat{F}^M \widehat{F}^N \quad (8.5)$$

is both closed, exact and δ exact:

$$d\mathcal{L}_T = \widehat{D}\mathcal{L}_T = 0, \quad \delta\mathcal{L}_T = 0, \quad (8.6)$$

$$\mathcal{L}_T = d(\widehat{A}\widehat{F} - 1/3\widehat{A}\widehat{A}\widehat{A}), \quad (8.7)$$

provided \widehat{g}_{MN} is the super-Killing metric of the super-algebra

$$\widehat{g}_{MN} = \frac{1}{2} STr(\Lambda_M \Lambda_N) = \frac{1}{2} Tr(\chi \Lambda_M \Lambda_N). \quad (8.8)$$

These assertions depend on the super-Jacobi identity (5.20)–(5.23) which implies that the quartic terms $\widehat{g}_{MN} (\widehat{A}\widehat{A})^M (\widehat{A}\widehat{A})^N$, hidden in the topological 4-form, actually vanish. Notice that the super-Killing metric \widehat{g}_{MN} (8.8) is skew for the odd indices

$$\widehat{g}_{ij} = -\widehat{g}_{ji}. \quad (8.9)$$

This anti-symmetry exactly fits the odd part of (8.5)

$$\widehat{F}^i \widehat{F}^j = -\Gamma\bar{\Gamma} D\Phi^i D\Phi^j = -\widehat{F}^j \widehat{F}^i, \quad (8.10)$$

whereas the even part of (8.5) is symmetric as usual

$$\widehat{g}_{ab} = \widehat{g}_{ba}, \quad \widehat{F}^a \widehat{F}^b = \widehat{F}^b \widehat{F}^a. \quad (8.11)$$

This concludes the study of the differential geometry of the chiral Yang-Mills theory. We will analyze why the model is interesting for particle physics in a separate paper.

9. Discussion

In Yang-Mills theory, the underlying symmetry group, which is truly the product of the $SO(3,1)$ Lorentz rotation group by the local gauge group, say $SU(2)U(1)$, is factorizable. The 2 indices a and μ of the Yang-Mills field A_μ^a do not speak to each other. Since the discovery by Ne'eman and Fairlie in 1979 of the $SU(2/1)$ structure of the weak interactions, many authors tried to integrate the super-algebra structure into Yang-Mills theory. The stumbling block, which prevented this fusion, was the implicit assumption that this new theory would also be factorizable. But mixing left and right spinors at a given point of space time breaks the Lorentz group. As we recalled in this paper, we can only mix left and right in a given direction of propagation x^μ and only using the σ matrices. In the same way, internal super-symmetry between particle with the same spin breaks the spin-statistics relation. For example, in 82 with Yuval Ne'eman [15], we tried to embed the $SU(2/1) Z_2$

grading in the Z grading of exterior forms, anticipating on the Quillen connection [12]. But this did not work because the left spin $1/2$ was rotated into a left spin $3/2$ state, and the non Abelian equations were not integrable.

The factorization issue was addressed by Connes and Lott in 1990 [2]. However, it is very difficult to build upon their foundations, because they seem to imply that the usual functional integral Lagrangian formalism of Quantum Field Theory must be replaced by a completely new framework based on a K -cycle and the Dixmier trace. Furthermore, their δ differential fails to close unless they quotient by their junk ideal which has no clear interpretation in QFT. Many papers followed which tried to explain the Connes-Lott formalism to the physicists by throwing away the Dirac matrices and reverting to the factorized framework. But, as shown by [11], they were throwing the baby with the bath and can be ignored.

The new construction proposed in this paper is clenching the rotation group and the internal super-symmetry in an indissoluble way. Neither the Dirac one form Γ (5.2), nor the odd matrix Λ_i (5.6) can act alone on the Fermions. Only their chiral product $\Gamma\Lambda_i$ is well defined and contribute to the connection one form (5.4) as $\Gamma\Lambda\Phi$. We then found that the covariant differential (5.10) is associative if and only if A^a, Φ^i is valued in the adjoint representation of a super-algebra (5.24). Then we showed that the super commutator with a fixed odd direction (6.1) implements the 2-point non-commutative differential of Connes-Lott (4.2) and leads as they found to spontaneous symmetry breakdown (6.15), but without leaving the standard framework of Quantum Field Theory. The super-Jacobi identity (5.23) automatically implements the consistency (6.17) of our definition (6.1) of the discrete differential δ . This construction is much simpler and completely different from [2], and leads to a similar but not identical expression for the Higgs potential and the choice of the $U(1)$ weak hyper-charge. If this chiral geometry really exists at very high energy, may be we should search for a compactification which would preserve down to 1000 Gev some of the left-right dialectic so characteristic of the 10-dimensional super-string.

Our new construction exactly fits the phenomenology of the weak interactions. If we grade the Fermions by their chirality (5.24), and consider the smallest simple super-algebra $SU(2/1)$, we recover the phenomenological model of Ne'eman and Fairlie [8, 6, 9]. This model appeared in 1979 as very promising. It requires that the weak $U(1)$ hyper-charge should be supertracless, automatically implementing the original choice of Weinberg [16] who wanted to avoid a current coupled to the lepton number, and explaining the non existence of any massless charged particle. In addition. the irreducible representations of $SU(2/1)$ naturally describe both the leptons [8, 6] and the quark [4, 10]. However no setting was found so far that could explain how to extend the Yang-Mills Lie algebra into a super-algebra, and no relation was found between chirality and the super-algebraic structure. We believe that the present paper exactly fills these gaps and will come back in a separate paper to the discussion of the $SU(2/1)$ model.

Acknowledgments

It is a pleasure to thank John Lott for his gentle guiding through non commutative geometry, Florian Scheck for an interesting reprint, Pierre Fayet and Victor Kac, my dependable

references in mathematics and physics, Gerard Menessier and Andre Neveu for many discussions and Yuval Ne'eman for our long lasting collaboration on $SU(2/1)$. We also thank David Lipman for inviting us to his wonderful laboratory and Danielle Thierry-Mieg for innumerable creative suggestions.

This research was supported in part by the Intramural Research Program of the NIH, National Library of Medicine.

References

- [1] C. Becchi, A. Rouet and R. Stora, *Renormalization of the Abelian Higgs-model*, *Commun. Math. Phys.* **42** (1975) 127.
- [2] A. Connes and J. Lott, *Particle models and noncommutative geometry*, *Nucl. Phys.* **18** (Proc. Suppl.) (1991) 29.
- [3] A. Connes and J. Lott, *The metric aspect of noncommutative geometry*, in *New symmetry principles in quantum field theory*, Cargèse, 1991, pp. 53–93, *NATO Adv. Sci. Inst. Ser. B Phys.*, 295, Plenum, New York, 1992.
- [4] P.H. Dondi and P.D. Jarvis, *A supersymmetric Weinberg-Salam model*, *Phys. Lett.* **B 84** (1979) 403.
- [5] L.D. Faddeev and V.N. Popov, *Feynman diagrams for the Yang-Mills field*, *Phys. Lett.* **B 25** (1967) 29.
- [6] D.B. Fairlie, *Higgs' fields and the determination of the Weinberg angle*, *Phys. Lett.* **B 82** (1979) 97.
- [7] Y. Ne'eman, *Derivation of strong interactions from a gauge invariance*, *Nucl. Phys.* **B 26** (1961) 222, reprinted in *The eightfold way*, p. 58 (in coll. with M. Gell-Mann) *Frontiers in physics* series, W.A. Benjamin, New York 1964.
- [8] Y. Neeman, *Irreducible gauge theory of a consolidated Weinberg-Salam model*, *Phys. Lett.* **B 81** (1979) 190.
- [9] Y. Neeman, S. Sternberg and D. Fairlie, *Superconnections for electroweak $SU(2/1)$ and extensions and the mass of the Higgs*, *Phys. Rept.* **406** (2005) 303.
- [10] Y. Ne'eman and J. Thierry-Mieg, *Geometrical gauge theory of ghost and Goldstone fields and of ghost symmetries*, *Proc. Nat. Acad. Sci.* **77** (1980) 720.
- [11] N. Papadopoulos, J. Plass and F. Scheck, *Models of electroweak interactions in non-commutative geometry, a comparison*, *Phys. Lett.* **B 324** (1994) 380 [[hep-th/9401136](#)]
- [12] D. Quillen, *Superconnections and the Chern character*, *Topology* **24** (1985) 89.
- [13] J. Thierry-Mieg, *Geometrical reinterpretation of Faddeev-Popov particles and BRS transformations*, *J. Math. Phys.* **21** (1980) 2834.
- [14] J. Thierry-Mieg, *Explicit classical construction of the Faddeev-Popov ghost fields*, *Nuovo Cim.* **56A** (1980) 396.
- [15] J. Thierry-Mieg and Y. Ne'eman, *Exterior gauging of an internal supersymmetry and $SU(2/1)$ quantum asthenodynamics*, *Proc. Nat. Acad. Sci.* **79** (1982) 7068.
- [16] S. Weinberg, *A model of leptons*, *Phys. Rev. Lett.* **19** (1967) 1264.
- [17] C.N. Yang and R. Mills, *Conservation of isotopic spin and isospin gauge invariance*, *Phys. Rev.* **96** (1954) 191.